1.31) a) $B_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, $B_2 = \left\{ \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \\ 6 \end{bmatrix} \right\}$

Puedo hallar $C_{B_1B_2}$ poniendo cada componente de B_1 como α de B_2.

$(1,0,0) = \alpha_1 . (1,3,-2) + \alpha_2 . (3,5,-6) + \alpha_3 . (0,-5,6)$ I

$(0,1,0) = \alpha_4 . (1,3,-2) + \alpha_5 . (3,5,-6) + \alpha_6 . (0,-5,6)$ II

$(0,0,1) = \alpha_7 . (1,3,-2) + \alpha_8 . (3,5,-6) + \alpha_9 . (0,-5,6)$ III

~~Se puede ver fácil que $\alpha_2 + \alpha_3 = 0$, $\alpha_3 = 0$, $\alpha_1 = 1$~~

~~Ecuaciones para I~~

ECUACIONES de I

$$\begin{cases} \alpha_1 + 3\alpha_2 = 1 \\ 3\alpha_1 + 5\alpha_2 - 5\alpha_3 = 0 \\ -2\alpha_1 - 6\alpha_2 + 6\alpha_3 = 0 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 0 & 1 \\ 3 & 5 & -5 & 0 \\ -2 & -6 & 6 & 0 \end{array}\right) \begin{array}{l} F_2 \to 3F_1 - F_2 \\ F_3 \to 2F_1 + F_3 \end{array} \quad \left(\begin{array}{ccc|c} 1 & 3 & 0 & 1 \\ 0 & 4 & 5 & 3 \\ 0 & 0 & 6 & 2 \end{array}\right)$$

$$\to \begin{cases} \alpha_1 + 3\alpha_2 = 1 \to \alpha_1 = 1 - 1 \to \boxed{\alpha_1 = 0} \\ 4\alpha_2 + 5\alpha_3 = 3 \to \alpha_2 = (3 - 5/3) \cdot 1/4 \to \boxed{\alpha_2 = \frac{1}{3}} \\ 6\alpha_3 = 2 \to \boxed{\alpha_3 = \frac{1}{3}} \end{cases}$$

Ecuaciones para II

$$\begin{cases} \alpha_4 + 3\alpha_5 = 0 \\ 3\alpha_4 + 5\alpha_5 - 5\alpha_6 = 1 \\ -2\alpha_4 - 6\alpha_5 + 6\alpha_6 = 0 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 3 & 5 & -5 & 1 \\ -2 & -6 & 6 & 0 \end{array}\right) \begin{array}{l} F2 \to 3F1 - F2 \\ F3 \to 2F1 + F3 \end{array} \left(\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 4 & 5 & -1 \\ 0 & 0 & 6 & 0 \end{array}\right)$$

$$\begin{cases} \alpha 4 + 3\alpha 5 = 0 \to \boxed{\alpha 4 = 3/4} \\ 4\alpha 5 + 5\alpha 6 = -1 \to \boxed{\alpha 5 = -1/4} \\ 6\alpha 6 = 0 \to \boxed{\alpha 6 = 0} \end{cases}$$

Ecuaciones p/ III

$$\begin{cases} \alpha 7 + 3\alpha 8 = 0 \\ 3\alpha 7 + 5\alpha 8 - 5\alpha 9 = 0 \\ -2\alpha 7 - 6\alpha 8 + 6\alpha 9 = 1 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 3 & 5 & -5 & 0 \\ -2 & -6 & 6 & 1 \end{array}\right) \begin{array}{l} F2 \to 3F1 - F2 \\ F3 \to 2F1 + F3 \end{array} \left(\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 0 & 6 & 1 \end{array}\right)$$

$$\to \alpha 7 + 3\alpha 8 = 0 \to \boxed{\alpha 7 = 5/8}$$

$$4\alpha 8 + 5\alpha 9 = 0 \to \boxed{\alpha 8 = -5/24}$$

$$6\alpha 9 = 1 \to \boxed{\alpha 9 = \frac{1}{6}}$$

Por lo tanto:

$$\boxed{C_{B1B2} = \begin{bmatrix} \alpha_1 & \alpha_4 & \alpha_7 \\ \alpha_2 & \alpha_5 & \alpha_8 \\ \alpha_3 & \alpha_6 & \alpha_9 \end{bmatrix} = \begin{bmatrix} 0 & 3/4 & 5/8 \\ 1/3 & -1/4 & -5/24 \\ 1/3 & 0 & 1/6 \end{bmatrix}}$$

Ahora, quiero $[v]_{B2}$, siendo $v = x_1 v_1 + x_2 v_2 + x_3 v_3$

Puedo obtenerlo con la siguiente fórmula:

$$[v]_{B2} = C_{B1B2} \cdot [v]_{B1}$$

En este caso $[v]_{B1} = [x_1 \ x_2 \ x_3]^T$

$$\rightarrow \boxed{[v]_{B2} = \begin{bmatrix} 0 & 3/4 & 5/8 \\ 1/3 & -1/4 & -5/24 \\ 1/3 & 0 & 1/6 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3/4\, x_2 + 5/8\, x_3 \\ 1/3\, x_1 - 1/4\, x_2 - 5/24\, x_3 \\ 1/3\, x_1 + 1/6\, x_3 \end{bmatrix}}$$

6) $B_1 = \{1, x, x^2\}$ base canónica de $\mathbb{R}_2[x]$

$B_2 = \{P_1, P_2, P_3\}$ base de $\mathbb{R}_2[x]$, siendo

$P_1 = \frac{x^2}{2} - \frac{3}{2}x + 1$, $P_2 = -x^2 + 2x$, $P_3 = \frac{x^2}{2} - \frac{x}{2}$

Como B_1 es la canónica, para hallar $C_{B_1B_2}$, puedo armar (fácilmente) $C_{B_2B_1}$ y hallar su inversa, que me dará $C_{B_1B_2}$.

$$C_{B_2B_1} = \begin{bmatrix} 1 & 0 & 0 \\ -3/2 & 2 & -1/2 \\ 1/2 & -1 & 1/2 \end{bmatrix}$$

Busco inversa.

$$(C_{B2B1})^{-1} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ -3/2 & 2 & -1/2 & 0 & 1 & 0 \\ 1/2 & -1 & 1/2 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} F2 \to 3/2F1 + F2 \\ F3 \to 1/2F1 - F3 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & -1/2 & 3/2 & 1 & 0 \\ 0 & 1 & -1/2 & 1/2 & 0 & -1 \end{array}\right) F3 \to F2 - 2F3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & -1/2 & 3/2 & 1 & 0 \\ 0 & 0 & 1/2 & 1/2 & 1 & 2 \end{array}\right) F2 \to F2 + F3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 2 & 2 & 2 \\ 0 & 0 & 1/2 & 1/2 & 1 & 2 \end{array}\right) \begin{array}{l} F2 \to F2/2 \\ F3 \to F3.2 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 4 \end{array}\right)$$

Por lo tanto $\boxed{C_{B1B2} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}} = (C_{B2B1})^{-1}$

Ahora quiero $[v]_{B2}$

Puedo usar la fórmula:

$$[v]_{B2} = C_{B1B2} \cdot [v]_{B1}$$

en donde $[v]_{B1} = \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix}$

$$\to [v]_{B2} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} \to \boxed{[v]_{B2} = \begin{bmatrix} x1 \\ x1+x2+x3 \\ x1+2x2+4x3 \end{bmatrix}}$$

c) $B_1 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$, canónica $\mathbb{R}^{2\times 2}$

$B_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$ base de $\mathbb{R}^{2\times 2}$

Como B_1 es la canónica, puedo hallar $C_{B_2B_1}$ fácilmente y calcular su inversa, que me dará $C_{B_1B_2}$.

$$C_{B_2B_1} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Busco inversa:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) F_3 \to F_3 - F_4 \quad \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) F_2 \to F_2 - F_3$$

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) F_2 \to F_2 - F_4 \quad \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) F_1 \to F_1 - F_4$$

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) F_1 \to F_1 - F_3 \quad \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) F_1 \to F_1 - F_2$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

Por lo tanto $\boxed{C_{B_1B_2} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}} = (C_{B_2B_1})^{-1}$

Ahora quiero $[v]_{B2}$:

Puedo usar la fórmula:

$$[v]_{B2} = C_{B1B2} \cdot [v]_{B1}$$

En este caso $[v]_{B1} = \begin{bmatrix} x1 \\ x2 \\ x3 \\ x4 \end{bmatrix}$

$$\rightarrow [v_{B2}] = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x1 \\ x2 \\ x3 \\ x4 \end{bmatrix} \rightarrow \boxed{[v_{B2}] = \begin{bmatrix} x1 - x2 \\ x2 - x3 \\ x3 - x4 \\ x4 \end{bmatrix}}$$